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A note on nonlinear stability of plane parallel shear flows ☆

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Abstract

We present a generalized energy functional $\mathcal{E}$ for plane parallel shear flows which provides conditional nonlinear stability for Reynolds numbers Re below some value $\mathrm{Re}_{\mathcal{E}}$ depending on the shear profile. In the case of the experimentally important profiles, viz. combinations of laminar Couette and Poiseuille flow, $\mathrm{Re}_{\mathcal{E}}$ is shown to be at least 174.

Keywords: Nonlinear stability; Parallel shear flows; Energy functional; Lyapunov method

1. Introduction

Although plane parallel shear flows of viscous incompressible fluids belong to the simplest hydrodynamical systems the stability of the basic flow is up to the present insufficiently understood. It is of particular interest to determine that value of the Reynolds number at which the onset of instability occurs. Most interesting are those shear flows

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which are of experimental relevance. These are the wall and pressure driven flows in plane parallel channels (Couette and Poiseuille flow, respectively) and linear combinations thereof.

The classical methods which yield rigorous stability results, are the energy method and the method of linearized stability. The former method yields global asymptotic stability for Reynolds numbers Re below some value $\mathrm{Re_E}$ which is of the order 10^2 for the above mentioned flows [1,8]. The second method yields a critical value $\mathrm{Re_c}$ below which the system is conditionally stable and above it is unstable. For Poiseuille flow there is $\mathrm{Re_c}$ of order 10^4, as numerically has been found (cf. [4]), whereas for Couette flow there is even $\mathrm{Re_c} = \infty$ [16]. Experimentally, the onset of instability is observed for Reynolds numbers of the order 10^3 (cf. [3,7]), i.e., none of the classical methods describes the stability behavior satisfactorily.

A third method which has successfully been applied to a couple of hydrodynamic stability problems uses generalized energy functionals $\mathcal{E}$ which are better adjusted to the specific problems under consideration [6,9,17]. This method of generalized energy functionals or Lyapunov direct method [5] provides in general conditional stability for Reynolds numbers below some value $\mathrm{Re}_{\mathcal{E}}$ together with explicit stability balls in the $\mathcal{E}^{1/2}$-norm. It has already been applied to plane parallel shear flows, however, under the assumption of stress-free boundary conditions for the perturbations [15]. Rigid boundary conditions, which are more appropriate, proved so far as a serious problem for the application of this method [11]. Only recently, using a refined calculus inequality, this problem could be resolved, and conditional nonlinear stability has been proved in the Couette flow case for Reynolds numbers below $\mathrm{Re}_{\mathcal{E}} = 177$ [12].

Here, conditional nonlinear stability is proved for arbitrary plane parallel shear flows up to some value $\mathrm{Re}_{\mathcal{E}}$ which depends on the shear profile. The corresponding functional $\mathcal{E}$ is simpler than that used in [12]. As a consequence $\mathrm{Re}_{\mathcal{E}}$ turns out to be $\mathrm{Re}_{\mathrm{E}}^{x}$, the ordinary energy stability limit for perturbations which do not vary in the spanwise direction. In the case of the experimentally important profiles, viz. linear combinations of Couette and Poiseuille flow, this number is at least 174, the value for pure Poiseuille flow. For Couette flow it coincides with the value obtained in [12].

2. Preliminaries

The appropriate geometrical setting for plane parallel shear flows is an infinite layer $\mathbb{R} \times (-\frac{1}{2}, \frac{1}{2})$ of thickness 1 with horizontal coordinates x, y and vertical coordinate z. Plane parallel shear flows are then characterized by the functional form

$$\mathbf{U}_0 = \mathbf{U}_0(z) = \mathrm{Re}\begin{pmatrix} f(z) \\ 0 \\ 0 \end{pmatrix}. \tag{2.1}$$

The function $f : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ is assumed to be sufficiently smooth and is called the shear profile. For Couette flow there is $f(z) = -z$ and for Poiseuille flow $f(z) = 1 - 4z^2$. $\mathrm{Re} > 0$

is the Reynolds number based on the distance between bottom and top boundaries of the layer and the maximum velocity difference in the flow. In order to investigate the stability of $\mathbf{U}_0$ we impose perturbations $\mathbf{u} = (u_x, u_y, u_z)$. These are governed by the system

$$\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} + \mathrm{Re}(f\partial_x \mathbf{u} + f' u_z \mathbf{e}_x) + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = 0, \\ \nabla\cdot\mathbf{u} = 0, \end{cases} \tag{2.2}$$

in $\mathbb{R}^2 \times (-\frac{1}{2}, \frac{1}{2}) \times (0, T)$, $T > 0$, and satisfy the boundary conditions

$$\mathbf{u}(x, y, z, t) = 0 \quad \text{for } (x, y, z) \in \mathbb{R}^2 \times \left\{-\frac{1}{2}, \frac{1}{2}\right\},\ t > 0. \tag{2.3}$$

Here $\mathbf{e}_x = (1, 0, 0)^{\mathrm{T}}$. The initial value $\mathbf{u}(\cdot, 0) = \mathbf{u}_0$ at time $t = 0$ is assumed to be given (and of course solenoidal). $\mathbf{u}$ corresponds to the velocity field of the perturbation and p denotes the pressure. Both $\mathbf{u}$ and ∇p are x, y-periodic with respect to a rectangle $\mathcal{P} = (-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}) \times (-\frac{\pi}{\beta}, \frac{\pi}{\beta})$ with wave numbers $(\alpha, \beta) \in \mathbb{R}^2_+$. In the following it suffices therefore to consider functions over the box

$$\Omega = \mathcal{P} \times \left(-\frac{1}{2}, \frac{1}{2}\right) = \left(-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}\right) \times \left(-\frac{\pi}{\beta}, \frac{\pi}{\beta}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right).$$

As basic function space we take $L^2(\Omega)$. In the sequel, $\|\cdot\|$ is always the norm in $L^2(\Omega)$ except in the case when applied to a function defined on $(-\frac{1}{2}, \frac{1}{2})$. Then, $\|\cdot\|$ means the norm in $L^2(-\frac{1}{2}, \frac{1}{2})$; the correct notion should be clear from the context. $(\cdot,\cdot)$ denotes always the scalar product associated with $\|\cdot\|$.

In order to cope with the divergence constraint $(2.2)_2$ we make use of the poloidal-toroidal decomposition [18]:

$$\mathbf{u} = \nabla \times \big(\nabla \times (\varphi\,\mathbf{e}_z)\big) + \nabla \times (\psi\,\mathbf{e}_z) + \mathbf{F} =: \boldsymbol{\delta}\varphi + \boldsymbol{\varepsilon}\psi + \mathbf{F}. \tag{2.4}$$

Here $\mathbf{e}_z = (0, 0, 1)^{\mathrm{T}}$. The functions φ and ψ are determined uniquely if one requires them to be periodic with respect to $\mathcal{P}$ and to fulfill $\int_{\mathcal{P}} \varphi(x, y, z)\,dx\,dy = \int_{\mathcal{P}} \psi(x, y, z)\,dx\,dy = 0$ for every $z \in (-\frac{1}{2}, \frac{1}{2})$. The first part in (2.4) is called the poloidal part of $\mathbf{u}$ and the second one the toroidal one. The third part, the mean flow, depends only on z and has constant third component. These three parts are mutually orthogonal in $L^2(\Omega)^3$. The vector operators $\boldsymbol{\delta}$ and $\boldsymbol{\varepsilon}$ have the form

$$\boldsymbol{\delta}\varphi = \begin{pmatrix} \partial_x\partial_z\varphi \\ \partial_y\partial_z\varphi \\ (-\Delta_2)\varphi \end{pmatrix}, \qquad \boldsymbol{\varepsilon}\psi = \begin{pmatrix} \partial_y\psi \\ -\partial_x\psi \\ 0 \end{pmatrix},$$

where $\Delta_2 = \partial_x^2 + \partial_y^2$ is the horizontal Laplacian. The boundary conditions (2.3) for $\mathbf{u}$ transform into

$$\varphi = \partial_z\varphi = 0, \quad \psi = 0, \quad F_x = F_y = 0 \quad \text{for } z = \pm\frac{1}{2}, \tag{2.5}$$

and $F_z(z) \equiv 0$. Applying the operators $\boldsymbol{\delta}$ and $\boldsymbol{\varepsilon}$ to Eq. $(2.2)_1$ as well as taking the mean with respect to $\mathcal{P}$ the system (2.2) can equivalently be formulated in terms of the new variables $(\varphi, \psi, F_x, F_y)$:

$$
\begin{aligned}
&(-\Delta)(-\Delta_2)\partial_t\varphi + \Delta^2(-\Delta_2)\varphi + \operatorname{Re} f(-\Delta)(-\Delta_2)\partial_x\varphi + \operatorname{Re} f''(-\Delta_2)\partial_x\varphi \\
&\quad + \boldsymbol{\delta}\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) = 0, \\
&(-\Delta_2)\partial_t\psi + (-\Delta)(-\Delta_2)\psi + \operatorname{Re} f(-\Delta_2)\partial_x\psi - \operatorname{Re} f'(-\Delta_2)\partial_y\varphi \\
&\quad - \boldsymbol{\varepsilon}\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) = 0, \\
&\partial_t F_x + \left(-\partial_z^2\right)F_x + \frac{1}{|\mathcal{P}|}\int\limits_{\mathcal{P}} \tilde{\mathbf{u}}\cdot\nabla\tilde{u}_x\,dx\,dy = 0, \\
&\partial_t F_y + \left(-\partial_z^2\right)F_y + \frac{1}{|\mathcal{P}|}\int\limits_{\mathcal{P}} \tilde{\mathbf{u}}\cdot\nabla\tilde{u}_y\,dx\,dy = 0.
\end{aligned}
\tag{2.6}
$$

$\tilde{\mathbf{u}} := \boldsymbol{\delta}\varphi + \boldsymbol{\varepsilon}\psi$ is that part of $\mathbf{u}$ which has vanishing mean value over $\mathcal{P}$ and $|\mathcal{P}| := \frac{4\pi^2}{\alpha\beta}$ denotes the volume of $\mathcal{P}$.

With $\Phi := (\varphi, \psi, F_x, F_y)^T$ a compact matrix notation can be used for system (2.6),

$$
\mathcal{B}\partial_t\Phi + \mathcal{A}\Phi - \operatorname{Re}\mathcal{C}\Phi + \mathcal{M}(\Phi,\Phi) = 0. \tag{2.7}
$$

Here, $\mathcal{B}$ and $\mathcal{A}$ are diagonal matrix operators, $\mathcal{C}$ is a nonnormal interaction matrix, and $\mathcal{M}$ is a bilinear form. The operator $\mathcal{A}$, for example, has the form

$$
\mathcal{A} = \operatorname{diag}\left(\Delta^2(-\Delta_2), (-\Delta)(-\Delta_2), \left(-\partial_z^2\right), \left(-\partial_z^2\right)\right),
$$

acting in the Hilbert space

$$
\mathcal{H} := L^2_M(\Omega)\times L^2_M(\Omega)\times L^2\left(\left(-\frac{1}{2},\frac{1}{2}\right)\right)\times L^2\left(\left(-\frac{1}{2},\frac{1}{2}\right)\right),
$$

where $L^2_M(\Omega)$ denotes the space $\{f \in L^2(\Omega) \mid \int_{\mathcal{P}} f(x,y,z)\,dx\,dy = 0 \text{ for a.e. } z \in (-\frac{1}{2},\frac{1}{2})\}$. The domain $D(\mathcal{A})$ is most easily described in terms of a Fourier mode expansion for φ and ψ with respect to the horizontal variables x and y,

$$
\varphi(x,y,z) = \frac{1}{\sqrt{|\mathcal{P}|}}\sum_{\boldsymbol{\kappa}\in\mathbb{Z}^2\setminus\{0\}} a_{\boldsymbol{\kappa}}(z)e^{i(\alpha\kappa_1 x + \beta\kappa_2 y)}, \tag{2.8}
$$

$$
\psi(x,y,z) = \frac{1}{\sqrt{|\mathcal{P}|}}\sum_{\boldsymbol{\kappa}\in\mathbb{Z}^2\setminus\{0\}} b_{\boldsymbol{\kappa}}(z)e^{i(\alpha\kappa_1 x + \beta\kappa_2 y)}. \tag{2.9}
$$

We then define (cf. [10,19])

$$
D(\mathcal{A}) = D\left(\Delta^2(-\Delta_2)\right)\times D\left((-\Delta)(-\Delta_2)\right)\times D\left(-\partial_z^2\right)\times D\left(-\partial_z^2\right),
$$

where

$$
\begin{aligned}
D\left(\Delta^2(-\Delta_2)\right) = \Bigg\{&\varphi \;\Big|\; \varphi \text{ expanded as in (2.8)}, \\
&a_{\boldsymbol{\kappa}} \in H^4\left(\left(-\frac{1}{2},\frac{1}{2}\right)\right),\ a_{\boldsymbol{\kappa}} = \partial_z a_{\boldsymbol{\kappa}} = 0 \text{ at } z = \pm\frac{1}{2},
\end{aligned}
$$

$$\sum_{\boldsymbol{\kappa}\in\mathbb{Z}^2\setminus\{0\}} \left(\alpha^2\kappa_1^2+\beta^2\kappa_2^2\right)^2 \int_{-1/2}^{1/2} \left|\left(-\partial_z^2+\alpha^2\kappa_1^2+\beta^2\kappa_2^2\right)^2 a_{\boldsymbol{\kappa}}(z)\right|^2 dz<\infty\Bigg\},$$

$$D\big((-\Delta)(-\Delta_2)\big)=\Bigg\{\psi\ \Big|\ \psi \text{ expanded as in (2.9)},$$

$$b_{\boldsymbol{\kappa}}\in H^2\left(\left(-\frac{1}{2},\frac{1}{2}\right)\right),\ b_{\boldsymbol{\kappa}}=0 \text{ at } z=\pm\frac{1}{2},$$

$$\sum_{\boldsymbol{\kappa}\in\mathbb{Z}^2\setminus\{0\}} \left(\alpha^2\kappa_1^2+\beta^2\kappa_2^2\right)^2 \int_{-1/2}^{1/2} \left|\left(-\partial_z^2+\alpha^2\kappa_1^2+\beta^2\kappa_2^2\right) b_{\boldsymbol{\kappa}}(z)\right|^2 dz<\infty\Bigg\},$$

and

$$D\left(-\partial_z^2\right)=H^2\left(\left(-\frac{1}{2},\frac{1}{2}\right)\right)\cap \mathring{H}^1\left(\left(-\frac{1}{2},\frac{1}{2}\right)\right).$$

With these definitions $\mathcal{A}$ is a self-adjoint and strictly positive operator. Thus, fractional powers of $\mathcal{A}$ make sense and can analogously be explained in terms of the expansions (2.8) and (2.9). Similar definitions apply to the operators $\mathcal{B}$ and $\mathcal{C}$.

The energy of the system (in the volume Ω) becomes in the new variables[1]

$$E=\frac{1}{2}\|\mathbf{u}\|^2=\frac{1}{2}\left\{\|\boldsymbol{\delta}\varphi\|^2+\|\boldsymbol{\varepsilon}\psi\|^2+|\mathcal{P}|\|\mathbf{F}\|^2\right\}, \tag{2.10}$$

and the variational expression determining Re_E takes the form (cf. [10])

$$\frac{|\Re(u_x, f'u_z)|}{\|\nabla\mathbf{u}\|^2}=\frac{|\Re((-\Delta_2)\varphi, f'(\partial_x\partial_z\varphi+\partial_y\psi+F_x))|}{\|(-\Delta)\boldsymbol{\varepsilon}\varphi\|^2+\|\boldsymbol{\delta}\psi\|^2+|\mathcal{P}|\|\partial_z\mathbf{F}\|^2}. \tag{2.11}$$

For later convenience we admit here complex valued velocity fields. Thus, the real part (denoted by $\Re$) of the interaction term appears in the numerator of (2.11). Re_E is then given by

$$\mathrm{Re}_\mathrm{E}^{-1}=\sup_{(\alpha,\beta)\in\mathbb{R}_+^2}\ \sup_{(\varphi,\psi)\in\mathcal{V}_{\alpha\beta}} \frac{|\Re((-\Delta_2)\varphi, f'(\partial_x\partial_z\varphi+\partial_y\psi))|}{\|(-\Delta)\boldsymbol{\varepsilon}\varphi\|^2+\|\boldsymbol{\delta}\psi\|^2}. \tag{2.12}$$

Note that $\mathbf{F}$ does not depend on x or y and, therefore, drops from the numerator of (2.11). Thus, $\mathbf{F}$ does not contribute to the supremum of (2.11) and can be omitted altogether.

The variational class $\mathcal{V}_{\alpha\beta}$ should reflect the mean value condition, the boundary conditions as well as the periodicity of the functions φ and ψ. Moreover, it should ensure that the supremum is in fact attained. A suitable choice is $\mathcal{V}_{\alpha\beta}=D(\tilde{\mathcal{A}}^{1/2})\setminus\{(0,0)\}$, where $\tilde{\mathcal{A}}$ is that part of $\mathcal{A}$ that is operating on (φ,ψ) in the Hilbert space $\tilde{\mathcal{H}}:=L_M^2(\Omega)\times L_M^2(\Omega)$.

If the class $\mathcal{V}_{\alpha\beta}$ of admissible functions is restricted to the class $\mathcal{V}_\alpha$ of functions depending only on x and z, or to the class $\mathcal{V}_\beta$ of functions depending only on y and z the

[1] We use the usual notation for L^2-scalar products of vector or tensor type quantities. Thus, there is, e.g., $\|\mathbf{u}\|^2=(\mathbf{u},\mathbf{u})=\sum_{i=1}^3(u_i,u_i)$ or $\|\nabla\mathbf{u}\|^2=(\nabla\mathbf{u},\nabla\mathbf{u})=\sum_{i,j=1}^3(\partial_i u_j,\partial_i u_j)$. Note that $\nabla\mathbf{u}$ is understood in the sense of a tensor product, whereas $\mathbf{u}\cdot\nabla=\sum_{i=1}^3 u_i\partial_i$ means the scalar product in $\mathbb{R}^3$.

corresponding 2-dimensional limits Re_E^x and Re_E^y are determined by the following simplified variational expressions:

$$1/\mathrm{Re}_\mathrm{E}^y = \sup_{\beta\in\mathbb{R}_+} \sup_{(\varphi,\psi)\in\mathcal{V}_\beta} \frac{|\Re((-\partial_y^2)\varphi,\, f'\partial_y\psi)|}{\|(-\partial_y^2-\partial_z^2)\partial_y\varphi\|^2 + \|\partial_y\partial_z\psi\|^2 + \|(-\partial_y^2)\psi\|^2}, \tag{2.13}$$

$$1/\mathrm{Re}_\mathrm{E}^x = \sup_{\alpha\in\mathbb{R}_+} \sup_{(\varphi,0)\in\mathcal{V}_\alpha} \frac{|\Re((-\partial_x^2)\varphi,\, f'\partial_x\partial_z\varphi)|}{\|(-\partial_x^2-\partial_z^2)\partial_x\varphi\|^2}. \tag{2.14}$$

It is well known that for Couette flow there is

$$\mathrm{Re}_\mathrm{E} = \mathrm{Re}_\mathrm{E}^y = 82.6\ldots, \qquad \mathrm{Re}_\mathrm{E}^x = 177\ldots, \tag{2.15}$$

and for Poiseuille flow

$$\mathrm{Re}_\mathrm{E} = \mathrm{Re}_\mathrm{E}^y = 99.1\ldots, \qquad \mathrm{Re}_\mathrm{E}^x = 174\ldots \tag{2.16}$$

(cf. [1,2,8,14]).

We conclude this section with a remark on the connection between the different representations for the flow field used here and in [15]. Formally, Eqs. $(2.6)_{1,2}$ are equivalent to Eqs. (3.4), (3.3) in [15] if one uses the correspondence

$$(-\Delta_2)\varphi \sim w, \qquad (-\Delta_2)\psi \sim \xi$$

between the poloidal/toroidal scalars φ and ψ and the "essential variables" w and ξ. Note, however, that the essential variables need not obey the mean-value-zero condition, which is obviously satisfied by $(-\Delta_2)\varphi$ and $(-\Delta_2)\psi$. Thus, the mean flow need not be treated separately and in this respect the flow field representation in [15] is simpler than here. The advantage of the poloidal–toroidal-mean flow representation consists in a one-to-one correspondence between $\mathbf{u}$ and $(\varphi, \psi, \mathbf{F})$ (cf. [18]), which is not as easy to establish between $\mathbf{u}$ and the essential variables.

3. Nonlinear stability

Let us consider the functional

$$\mathcal{E} = \mathcal{E}_1[\varphi] + \mathcal{E}_2[\mathbf{u},\mathbf{F}] := \frac{1}{2}\|\boldsymbol{\delta}\varphi\|^2 + \frac{1}{2}\big\{\sigma\|\boldsymbol{\varepsilon}\mathbf{u}\|^2 + \rho\,|\mathcal{P}|\,\|\mathbf{F}\|^2\big\}, \tag{3.1}$$

where the nonnegative coupling constants σ and ρ have yet to be fixed. We determine a generalized energy limit $\mathrm{Re}_\mathcal{E}$ with just $\mathcal{E}_1$, and use $\mathcal{E}_2$ to dominate the nonlinear term arising in the (generalized) energy balance of $\mathcal{E}_1$.

Multiplying $(2.6)_1$ with φ and integrating over Ω one obtains after partial integration and use of the boundary conditions (2.5) the generalized energy balance for $\mathcal{E}_1$,

$$\partial_t\mathcal{E}_1 = -\mathcal{D}_1 + \mathrm{Re}\,\mathcal{I}_1 + \mathcal{N}_1 \tag{3.2}$$

with

$$\begin{aligned} &\mathcal{D}_1[\varphi] := \big\|(-\Delta)\boldsymbol{\varepsilon}\varphi\big\|^2, \qquad \mathcal{I}_1[\varphi] := \Re\big((-\Delta_2)\varphi,\, f'\partial_x\partial_z\varphi\big),\\ &\mathcal{N}_1[\varphi,\psi,\mathbf{F}] := -\Re\big((\mathbf{u}\cdot\nabla\mathbf{u}),\, \boldsymbol{\delta}\varphi\big). \end{aligned} \tag{3.3}$$

The generalized energy limit $\mathrm{Re}_{\mathcal{E}}$ is then determined by

$$\mathrm{Re}_{\mathcal{E}}^{-1} = \sup_{(\alpha,\beta)\in\mathbb{R}_+^2} \sup_{\varphi\in\mathcal{W}_{\alpha\beta}} \frac{\mathcal{I}_1}{\mathcal{D}_1}[\varphi] \tag{3.4}$$

with

$$\frac{\mathcal{I}_1}{\mathcal{D}_1}[\varphi] = \frac{|\Re((-\Delta_2)\varphi, f'\partial_x\partial_z\varphi)|}{\|(-\Delta)\boldsymbol{\varepsilon}\varphi\|^2} \tag{3.5}$$

and $\mathcal{W}_{\alpha\beta} := D(\Delta^2(-\Delta_2)) \setminus \{0\}$. Note that in (3.5) $\mathcal{I}_1$ can always be replaced by $|\mathcal{I}_1|$ as with $\varphi(x,y,z) \in \mathcal{W}_{\alpha\beta}$, $\varphi(-x,-y,z)$ is also admissible. Thus, $\mathcal{I}_1$ can always be chosen positive without affecting $\mathcal{D}_1$.

The following proposition identifies $\mathrm{Re}_{\mathcal{E}}$ with $\mathrm{Re}_{\mathrm{E}}^x$.

Proposition 1. *Let* $\mathrm{Re}_{\mathcal{E}}$ *be the* 3*-dimensional* (*generalized*) *energy stability limit* (3.4) *corresponding to the functional*

$$\mathcal{E}_1[\varphi] = \frac{1}{2}\|\boldsymbol{\delta}\varphi\|^2,$$

and $\mathrm{Re}_{\mathrm{E}}^x$ *the* 2*-dimensional* (*ordinary*) *energy stability limit* (2.14) *taken with respect to functions which depend only on* x *and* z*. Then, there holds*

$$\mathrm{Re}_{\mathcal{E}} = \mathrm{Re}_{\mathrm{E}}^x.$$

Proof. Restricting φ to y-independent functions $\varphi(x,z)$ in the variational expression (3.5) yields

$$\frac{\mathcal{I}_1}{\mathcal{D}_1}[\varphi] = \frac{|\Re((-\Delta_2)\varphi, f'\partial_x\partial_z\varphi)|}{\|(-\Delta)\boldsymbol{\varepsilon}\varphi\|^2} = \frac{|\Re((-\partial_x^2)\varphi, f'\partial_x\partial_z\varphi)|}{\|(-\partial_x^2-\partial_z^2)\partial_x\varphi\|^2},$$

which is the variational expression corresponding to $\mathrm{Re}_{\mathrm{E}}^x$. This implies $\mathrm{Re}_{\mathcal{E}} \leqslant \mathrm{Re}_{\mathrm{E}}^x$.

In order to prove the converse inequality let us decompose $\varphi \in \mathcal{W}_{\alpha\beta}$ into its even and odd components with respect to the variable x, i.e.,

$$\varphi = \varphi_e + \varphi_o \tag{3.6}$$

with

$$\varphi_e(-x,y,z) = \varphi_e(x,y,z), \qquad \varphi_o(-x,y,z) = -\varphi_o(x,y,z).$$

Inserting (3.6) into (3.5) yields

$$\begin{aligned}\frac{|\Re((-\Delta_2)\varphi, f'\partial_x\partial_z\varphi)|}{\|(-\Delta)\boldsymbol{\varepsilon}\varphi\|^2} &= \frac{|\Re((-\Delta_2)\varphi_e, f'\partial_x\partial_z\varphi_o) + \Re((-\Delta_2)\varphi_o, f'\partial_x\partial_z\varphi_e)|}{\|(-\Delta)\boldsymbol{\varepsilon}\varphi_e\|^2 + \|(-\Delta)\boldsymbol{\varepsilon}\varphi_o\|^2} \\ &\leqslant \frac{|\Re((-\Delta_2)\varphi_e, f'\partial_x\partial_z\varphi_o) + \Re((-\Delta_2)\varphi_o, f'\partial_x\partial_z\varphi_e)|}{\|(-\Delta)\partial_x\varphi_e\|^2 + \|(-\Delta)\boldsymbol{\varepsilon}\varphi_o\|^2} =: \mathcal{F}[\varphi_e,\varphi_o].\end{aligned} \tag{3.7}$$

We have, therefore,

$$\sup_{(\alpha,\beta)\in\mathbb{R}_+^2} \sup_{\varphi\in\mathcal{W}_{\alpha\beta}} \frac{\mathcal{I}_1}{\mathcal{D}_1}[\varphi] \leqslant \sup_{(\alpha,\beta)\in\mathbb{R}_+^2} \sup_{(\varphi_e,\varphi_o)\in\tilde{\mathcal{W}}_{\alpha\beta}} \mathcal{F}[\varphi_e,\varphi_o], \tag{3.8}$$

where

$$\tilde{\mathcal{W}}_{\alpha\beta} := \big\{(\varphi_e, \varphi_o) \mid \varphi_e \in \mathcal{W}_{\alpha\beta} \text{ even in } x, \varphi_o \in \mathcal{W}_{\alpha\beta} \text{ odd in } x\big\}.$$

We are going to prove now

$$\sup_{(\alpha,\beta)\in\mathbb{R}_+^2} \sup_{(\varphi_e,\varphi_o)\in\tilde{\mathcal{W}}_{\alpha\beta}} \mathcal{F}[\varphi_e,\varphi_o] = \sup_{\alpha\in\mathbb{R}_+} \sup_{(\varphi,0)\in\mathcal{V}_\alpha} \frac{|\Re((-\partial_x^2)\varphi, f'\partial_x\partial_z\varphi)|}{\|(-\partial_x^2-\partial_z^2)\partial_x\varphi\|^2}. \tag{3.9}$$

Let $(\alpha,\beta)\in\mathbb{R}_+^2$ and $(\varphi_e^m,\varphi_o^m)\in\tilde{\mathcal{W}}_{\alpha\beta}$ such that $(\varphi_e^m,\varphi_o^m)$ maximizes $\mathcal{F}$. φ_e^m, φ_o^m satisfy then the Euler–Lagrange equations associated with $\mathcal{F}$,

$$\begin{aligned}
&\Delta^2\big(-\partial_x^2\big)\varphi_e - \frac{1}{2}\mu\big(2f'(-\Delta_2)\partial_x\partial_z\varphi_o + f''(-\Delta_2)\partial_x\varphi_o\big) = 0,\\
&\Delta^2(-\Delta_2)\varphi_o - \frac{1}{2}\mu\big(2f'(-\Delta_2)\partial_x\partial_z\varphi_e + f''(-\Delta_2)\partial_x\varphi_e\big) = 0,
\end{aligned} \tag{3.10}$$

together with the boundary conditions

$$\varphi_e = \partial_z\varphi_e = \varphi_o = \partial_z\varphi_o = 0 \quad \text{at } z = \pm\frac{1}{2}. \tag{3.11}$$

Due to the maximum property of $(\varphi_e^m,\varphi_o^m)$, μ is the smallest positive eigenvalue of the problem (3.10), (3.11), in particular, there is $\mu = \mathrm{Re}_{\mathcal{E}}$. Inserting appropriate mode expansions for φ_e and φ_o,

$$\begin{aligned}
\varphi_e(x,y,z) &= \sum_{\boldsymbol{\kappa}\in\mathbb{N}_0\times\mathbb{Z}\setminus\{(0,0)\}} a_{\boldsymbol{\kappa}}^e(z)\cos\alpha\kappa_1 x e^{i\beta\kappa_2 y},\\
\varphi_o(x,y,z) &= \sum_{\boldsymbol{\kappa}\in\mathbb{N}_0\times\mathbb{Z}\setminus\{(0,0)\}} a_{\boldsymbol{\kappa}}^o(z)\sin\alpha\kappa_1 x e^{i\beta\kappa_2 y}
\end{aligned} \tag{3.12}$$

into (3.10) we obtain

$$\begin{aligned}
&\big(A_{\boldsymbol{\kappa}}^2-\partial_z^2\big)^2\alpha^2\kappa_1^2 a_\kappa^e - \frac{1}{2}\mu\big(2f'A_{\boldsymbol{\kappa}}^2\alpha\kappa_1\partial_z a_\kappa^o + f''A_{\boldsymbol{\kappa}}^2\alpha\kappa_1 a_\kappa^o\big) = 0,\\
&\big(A_{\boldsymbol{\kappa}}^2-\partial_z^2\big)^2 A_{\boldsymbol{\kappa}}^2 a_\kappa^o + \frac{1}{2}\mu\big(2f'A_{\boldsymbol{\kappa}}^2\alpha\kappa_1\partial_z a_\kappa^e + f''A_{\boldsymbol{\kappa}}^2\alpha\kappa_1 a_\kappa^e\big) = 0, \quad \boldsymbol{\kappa}\in\mathbb{N}\times\mathbb{Z},
\end{aligned} \tag{3.13}$$

with $A_{\boldsymbol{\kappa}} := \sqrt{\alpha^2\kappa_1^2+\beta^2\kappa_2^2}$ and each $a_{\boldsymbol{\kappa}}^{e/o}$ satisfying the boundary conditions $a_{\boldsymbol{\kappa}}^{e/o} = \partial_z a_{\boldsymbol{\kappa}}^{e/o} = 0$ at $z = \pm 1/2$.

Observe now that the maximum of $\mathcal{F}$ is obtained by a single mode with (say) mode number $\hat{\boldsymbol{\kappa}}$ in the expansion (3.12), as can be seen as follows: Assume the maximum is attained by a (possibly infinite) linear combination of modes. Inserting this combination in $\mathcal{F}$ the numerator as well as the denominator decompose in a sum of bilinear terms each containing a single mode. Without restriction the modes can be chosen such that the expansion of the numerator contains only nonnegative terms. Applying Lemma A.1 (cf. Appendix A) we can select a single mode with maximal ratio, which at most increases the value of the variational expression.

Moreover, there is $\hat{\kappa}_1 > 0$ since all modes with mode numbers of the form $(0, \kappa_2)$ drop from the numerator of $\mathcal{F}$. Defining

$$c(z) := \alpha^2 \hat{\kappa}_1^2 a_{\hat{\kappa}}^e(z), \quad \tilde{c}(z) := \alpha \hat{\kappa}_1 a_{\hat{\kappa}}^o(z), \quad \gamma := A_{\hat{\boldsymbol{\kappa}}},$$

we obtain, therefore, a nontrivial solution of the system

$$\begin{aligned}&\left(\gamma^2 - \partial_z^2\right)^2 \gamma c - \frac{1}{2}\mu\left(2f'\gamma^3\partial_z\tilde{c} + f''\gamma^3\tilde{c}\right) = 0,\\&\left(\gamma^2 - \partial_z^2\right)^2 \gamma^2 \tilde{c} + \frac{1}{2}\mu\left(2f'\gamma^2\partial_z c + f''\gamma^2 c\right) = 0.\end{aligned}$$

Defining, furthermore,

$$\chi_e(x,z) = \frac{1}{\gamma}c(z)\cos\gamma x, \qquad \chi_o(x,z) = \tilde{c}(z)\sin\gamma x,$$

we end up with a solution of the system

$$\begin{aligned}&\left(-\partial_x^2 - \partial_z^2\right)^2\left(-\partial_x^2\right)\chi_e - \frac{1}{2}\mu\left(2f'\left(-\partial_x^2\right)\partial_x\partial_z\chi_o + f''\left(-\partial_x^2\right)\partial_x\chi_o\right) = 0,\\&\left(-\partial_x^2 - \partial_z^2\right)^2\left(-\partial_x^2\right)\chi_o - \frac{1}{2}\mu\left(2f'\left(-\partial_x^2\right)\partial_x\partial_z\chi_e + f''\left(-\partial_x^2\right)\partial_x\chi_e\right) = 0\end{aligned} \tag{3.14}$$

satisfying the boundary conditions

$$\chi_e = \partial_z\chi_e = \chi_o = \partial_z\chi_o = 0 \quad \text{at } z = \pm\frac{1}{2},$$

and being periodic in x with periodicity length $2\pi/\gamma$. Observe now that $(\chi, 0)$ with $\chi := \chi_e + \chi_o$ is in the variational class $\mathcal{V}_\alpha$ with $\alpha = \gamma$. Inserting χ into the variational expression in (2.14) yields with (3.14),

$$\frac{|\Re((-\partial_x^2)\chi, f'\partial_x\partial_z\chi)|}{\|(-\partial_x^2 - \partial_z^2)\partial_x\chi\|^2} = \frac{|\Re((-\partial_x^2)\chi_e, f'\partial_x\partial_z\chi_o) + \Re((-\partial_x^2)\chi_o, f'\partial_x\partial_z\chi_e)|}{\|(-\partial_x^2 - \partial_z^2)\partial_x\chi_e\|^2 + \|(-\partial_x^2 - \partial_z^2)\partial_x\chi_o\|^2} = \frac{1}{\mu}.$$

Thus we can conclude $1/\mathrm{Re}_\mathrm{E}^x \geqslant 1/\mu = 1/\mathrm{Re}_\mathcal{E}$. □

For arbitrary shear flows it is not quite clear whether

$$\mathrm{Re}_\mathrm{E}^x > \mathrm{Re}_\mathrm{E} \tag{3.15}$$

always holds. There are related results which estimate Re_E in terms of Re_E^x and Re_E^y (cf. [2] and [10]) and weaker estimates, e.g., $\mathrm{Re}_\mathrm{E}^x \geqslant 16/27\,\mathrm{Re}_\mathrm{E}^y$ [10], but no general result of the type (3.15).

If no external body forces are allowed it is well known that the most general shear profile compatible with viscous flow is a second order polynomial. For the most prominent examples, Couette and Poiseuille flow, (3.15) is clearly satisfied (cf. Eqs. (2.15), (2.16)). More generally, for any combination of Couette and Poiseuille flow, we have

Proposition 2. *Let $f_C = -z$ and $f_P = 1 - 4z^2$ be the Couette and Poiseuille profile, respectively. Let, furthermore $f_q := qf_C + (1-q)f_P$ with $q \in [0,1]$, and $\mathrm{Re}_\mathrm{E}(q)$ the energy stability limit for the profile f_q. Then,*

$$\mathrm{Re}_\mathrm{E}^x(q) \geqslant \min\bigl\{\mathrm{Re}_\mathrm{E}^x(0), \mathrm{Re}_\mathrm{E}^x(1)\bigr\} \geqslant 174\ldots \tag{3.16}$$

holds for any $q \in [0, 1]$.

Proof. Applying Lemma A.1 of Appendix A on the variational expression of Re_E^x we obtain

$$\begin{aligned}\frac{|\Re((-\partial_x^2)\varphi, f_q'\partial_x\partial_z\varphi)|}{\|(-\partial_x^2-\partial_z^2)\partial_x\varphi\|^2} &\leqslant \frac{q|\Re((-\partial_x^2)\varphi, f_C'\partial_x\partial_z\varphi)| + (1-q)|\Re((-\partial_x^2)\varphi, f_P'\partial_x\partial_z\varphi)|}{(q+(1-q))\|(-\partial_x^2-\partial_z^2)\partial_x\varphi\|^2} \\ &\leqslant \max\left\{\frac{|\Re((-\partial_x^2)\varphi, f_C'\partial_x\partial_z\varphi)|}{\|(-\partial_x^2-\partial_z^2)\partial_x\varphi\|^2}, \frac{|\Re((-\partial_x^2)\varphi, f_P'\partial_x\partial_z\varphi)|}{\|(-\partial_x^2-\partial_z^2)\partial_x\varphi\|^2}\right\}.\end{aligned}$$

Thus, taking the maximum with respect to $\varphi \in \mathcal{W}_\alpha$, $\alpha \in \mathbb{R}_+$, where $\mathcal{W}_\alpha$ is again the restriction of $\mathcal{W}_{\alpha\beta}$ to functions depending only on x and z, we have

$$\frac{1}{\mathrm{Re}_\mathrm{E}^x(q)} \leqslant \max\left\{\frac{1}{\mathrm{Re}_\mathrm{E}^x(0)}, \frac{1}{\mathrm{Re}_\mathrm{E}^x(1)}\right\}, \quad q \in [0, 1],$$

and with (2.15), (2.16),

$$\mathrm{Re}_\mathrm{E}^x(q) \geqslant 174\ldots, \quad q \in [0, 1]. \quad \square$$

On the other hand, inserting the maximizing solution (φ, ψ) for Couette flow and Poiseuille flow, respectively, into the expression

$$\frac{|\Re((-\partial_y^2)\varphi, (qf_C' + (1-q)f_P')\partial_y\psi)|}{\|(-\partial_y^2-\partial_z^2)\partial_y\varphi\|^2 + \|\partial_y\partial_z\psi\|^2 + \|(-\partial_y^2)\psi\|^2},$$

and observing that φ, ψ are both even with respect to z for Couette flow but even and odd for Poiseuille flow, one obtains with (2.13) and (2.15), (2.16),

$$\mathrm{Re}_\mathrm{E}(q) \leqslant \mathrm{Re}_\mathrm{E}^y(q) \leqslant \min\left\{\frac{1}{q}82.6\ldots, \frac{1}{1-q}99.1\ldots\right\}. \tag{3.17}$$

So, concerning the question asked above, it follows from Eqs. (3.16) and (3.17) that inequality (3.15) holds at least for $q \in [0, 0.43] \cup [0.47, 1]$.

We have now the following stability result.

Theorem 3. *Let* $(\varphi, \psi, \mathbf{F})$ *be a perturbation of the basic flow* $\mathbf{U}_0 = \mathrm{Re}(f(z), 0, 0)^T$ *with arbitrary shear profile* $f(z)$ *satisfying the system* (2.6) *under rigid boundary conditions* (2.5) *and being periodic in the horizontal variables* x, y *with wave numbers* $(\alpha, \beta) \in \mathbb{R}_+^2$. *Let* $0 < \mathrm{Re} < \mathrm{Re}_\mathcal{E} = \mathrm{Re}_\mathrm{E}^x$, $\Delta\mathrm{Re} := 1 - \mathrm{Re}/\mathrm{Re}_\mathcal{E}$, $C := 8(\sqrt{2}/m)^{3/2}$ *and* $m := \min(\alpha, \beta)$. *Moreover, let*

$$\mathcal{E}[\varphi, \psi, \mathbf{F}] = \frac{1}{2}\left\{\|\boldsymbol{\delta}\varphi\|^2 + \sigma\|\boldsymbol{\varepsilon}\mathbf{u}\|^2 + \rho\frac{4\pi^2}{\alpha\beta}\|\mathbf{F}\|^2\right\}$$

be a generalized energy functional with coupling parameters

$$\sigma = \frac{\pi^2\Delta\mathrm{Re}}{\mathrm{Re}^2}, \qquad \rho = \Delta\mathrm{Re}\frac{\alpha\beta m^3}{\sqrt{2}2^7\pi\mathrm{Re}^2}. \tag{3.18}$$

Then, any solution $(\varphi, \psi, \mathbf{F})$ *of* (2.5), (2.6) *decays in the norm* $\mathcal{E}^{1/2}$ *exponentially to zero,*

$$\mathcal{E}(t) \leqslant \mathcal{E}(0) \exp\bigl\{-\pi^2 \Delta\mathrm{Re}\bigl[1 - \bigl(\mathcal{E}(0)/\delta\bigr)^{1/2}\bigr]t\bigr\}, \tag{3.19}$$

provided the initial value satisfies

$$\mathcal{E}(0) < \delta := \frac{\sigma}{8C^2}\left(1 + \sqrt{\rho} + \sqrt{\frac{1}{\sigma m^2} + \frac{1}{\rho}}\right)^{-2}. \tag{3.20}$$

Proof. The second part $\mathcal{E}_2$ of the functional (3.1) is the same as in [12], the estimates, however, differ slightly. Special attention has to be paid only where $\mathcal{E}_1$ or $\mathcal{D}_1$ enter the estimates. So, we are brief where we just repeat calculations of [12].

By scalar multiplication of Eq. (2.2) with $\sigma \Delta_2 \mathbf{u}$ and of Eqs. $(2.6)_{3,4}$ with ρF_x, ρF_y and using (2.3), (2.5) we arrive at the energy balance for $\mathcal{E}_2$,

$$\partial_t \mathcal{E}_2 = -\mathcal{D}_2 + \mathrm{Re}\,\mathcal{I}_2 + \mathcal{N}_2, \tag{3.21}$$

where

$$\begin{aligned}
&\mathcal{D}_2[\mathbf{u}, \mathbf{F}] = \sigma \|\boldsymbol{\delta}\mathbf{u}\|^2 + \rho|\mathcal{P}|\|\mathbf{F}'\|^2,\\
&\mathcal{I}_2[\mathbf{u}, \mathbf{F}] = \sigma \Re(\boldsymbol{\varepsilon} u_z, \boldsymbol{\varepsilon} u_x),\\
&\mathcal{N}_2[\mathbf{u}, \mathbf{F}] = -\sigma \Re(\boldsymbol{\varepsilon}\mathbf{u} \cdot \nabla \mathbf{u}, \boldsymbol{\varepsilon}\mathbf{u}) - \rho \Re(\tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}}, \mathbf{F}).
\end{aligned} \tag{3.22}$$

With $\Delta\mathrm{Re} := 1 - \mathrm{Re}/\mathrm{Re}_{\mathcal{E}}$ and $\mathcal{D} := \Delta\mathrm{Re}\,\mathcal{D}_1 + \mathcal{D}_2$, the interaction term $\mathrm{Re}\,\mathcal{I}_2$ can be estimated as in [12]:

$$\begin{aligned}
\mathrm{Re}\,\mathcal{I}_2 &\leqslant \mathrm{Re}\,\sigma\bigl|(\boldsymbol{\varepsilon} u_z, \boldsymbol{\varepsilon} u_x)\bigr| \leqslant \mathrm{Re}\,\sigma^{1/2}\bigl\|\boldsymbol{\varepsilon}(-\Delta_2)\varphi\bigr\|\sigma^{1/2}\|\boldsymbol{\varepsilon}\mathbf{u}\| \leqslant \mathrm{Re}\,\sigma^{1/2}\mathcal{D}_1^{1/2}(2\mathcal{E}_2)^{1/2}\\
&\leqslant (\Delta\mathrm{Re})^{1/2}\mathcal{D}_1^{1/2}\mathcal{D}_2^{1/2} \leqslant \frac{1}{2}(\Delta\mathrm{Re}\,\mathcal{D}_1 + \mathcal{D}_2) = \frac{1}{2}\mathcal{D}.
\end{aligned} \tag{3.23}$$

Here, we used the estimate $2\mathcal{E}_2 \leqslant \mathcal{D}_2/\pi^2$, which follows with (A.2) and σ from (3.18).

$\mathcal{N}_1$ is splitted in 3 factors:

$$\mathcal{N}_1 \leqslant \bigl|(\mathbf{u} \cdot \nabla \mathbf{u}, \boldsymbol{\delta}\varphi)\bigr| \leqslant \operatorname*{ess\,sup}_{\Omega} |\mathbf{u}|\, \|\nabla \mathbf{u}\|\, \|\boldsymbol{\delta}\varphi\|.$$

With (3.18) and (A.7) we obtain for the first factor

$$\begin{aligned}
\|\mathbf{u}\|_\infty &\leqslant \frac{C}{\sqrt{2}}\|\boldsymbol{\delta}\tilde{\mathbf{u}}\| + \sqrt{\frac{2}{\pi}}\|\mathbf{F}'\| \leqslant \frac{C}{\sqrt{2\sigma}}\bigl\{\sqrt{\sigma}\,\|\boldsymbol{\delta}\tilde{\mathbf{u}}\| + \bigl(\rho|\mathcal{P}|\bigr)^{1/2}\|\mathbf{F}'\|\bigr\}\\
&\leqslant \frac{C}{\sqrt{\sigma}}\mathcal{D}_2^{1/2} \leqslant \frac{C}{\sqrt{\sigma}}\mathcal{D}^{1/2}
\end{aligned}$$

with $C = 8(\sqrt{2}/m)^{3/2}$ and $m = \min\{\alpha, \beta\}$. With

$$\|\nabla\tilde{\mathbf{u}}\| \leqslant \frac{1}{m}\|\boldsymbol{\varepsilon}\nabla\tilde{\mathbf{u}}\| \leqslant \frac{1}{m\sqrt{\sigma}}\mathcal{D}_2^{1/2}$$

we obtain for the second factor

$$\|\nabla \mathbf{u}\| = \bigl(\|\nabla\tilde{\mathbf{u}}\|^2 + |\mathcal{P}|\,\|\mathbf{F}'\|^2\bigr)^{1/2} \leqslant \bigl(\mathcal{D}_2/(\sigma m^2) + \mathcal{D}_2/\rho\bigr)^{1/2} \leqslant \sqrt{\frac{1}{\sigma m^2} + \frac{1}{\rho}}\,\mathcal{D}^{1/2},$$

and for the third:

$$\|\boldsymbol{\delta}\varphi\| \leqslant \sqrt{2}\,\mathcal{E}^{1/2}.$$

Thus, we end up with

$$\mathcal{N}_1 \leqslant \frac{\sqrt{2}C}{\sqrt{\sigma}}\sqrt{\frac{1}{\sigma m^2}+\frac{1}{\rho}}\,\mathcal{D}\mathcal{E}^{1/2}. \tag{3.24}$$

$\mathcal{N}_2$ is estimated as in [12]:

$$\begin{aligned}\mathcal{N}_2 &\leqslant \sigma\|\mathbf{u}\|_\infty\|\boldsymbol{\delta}\mathbf{u}\|\,\|\boldsymbol{\varepsilon}\mathbf{u}\| + \rho\|\tilde{\mathbf{u}}\|_\infty\,|\mathcal{P}|^{1/2}\|\mathbf{F}'\|\,\|\tilde{u}_z\| \\ &\leqslant \frac{C}{\sqrt{\sigma}}\,\mathcal{D}^{1/2}\mathcal{D}_2^{1/2}(2\mathcal{E}_2)^{1/2} + \sqrt{\rho}\frac{C}{\sqrt{\sigma}}\,\mathcal{D}^{1/2}\mathcal{D}_2^{1/2}(2\mathcal{E}_1)^{1/2} \\ &\leqslant \frac{\sqrt{2}\,C}{\sqrt{\sigma}}\,(1+\sqrt{\rho}\,)\,\mathcal{D}\mathcal{E}^{1/2}. \end{aligned} \tag{3.25}$$

Summarizing (3.24) and (3.25) we have

$$\mathcal{N}_1 + \mathcal{N}_2 \leqslant \frac{1}{2}\mathcal{D}(\mathcal{E}/\delta)^{1/2} \tag{3.26}$$

with δ given in (3.20).

Finally, we add up equations (3.2) and (3.21), apply Proposition 1, and use the estimates (3.23) and (3.26). This yields

$$\begin{aligned}\partial_t\mathcal{E} &= -\big[\mathcal{D}_1(1-\mathrm{Re}\,\mathcal{I}_1/\mathcal{D}_1)+\mathcal{D}_2\big] + \mathrm{Re}\,\mathcal{I}_2 + \mathcal{N}_1 + \mathcal{N}_2 \\ &\leqslant -\mathcal{D} + \frac{1}{2}\mathcal{D} + \frac{1}{2}\mathcal{D}(\mathcal{E}/\delta)^{1/2} \leqslant -\frac{1}{2}\mathcal{D}\big[1-(\mathcal{E}/\delta)^{1/2}\big] \\ &\leqslant -\frac{1}{2}\mathcal{D}\big[1-\big(\mathcal{E}(0)/\delta\big)^{1/2}\big] \leqslant -\pi^2\Delta\mathrm{Re}\,\mathcal{E}\big[1-\big(\mathcal{E}(0)/\delta\big)^{1/2}\big]. \end{aligned} \tag{3.27}$$

In the last line we used that $\mathcal{E}(t)$ is monotonically nonincreasing if $\mathcal{E}(0) < \delta$ and that $\mathcal{D} \geqslant 2\pi^2\Delta\mathrm{Re}\,\mathcal{E}$, which follows from the inequalities (A.2), (A.3), and $0 < \Delta\mathrm{Re} < 1$. Integrating (3.27) yields then (3.19). □

We close with two remarks:

(1) The functional $\mathcal{E}$ dominates the classical energy $E = \frac{1}{2}\|\mathbf{u}\|^2$. The relevant constant depends, however, on m and $\Delta\mathrm{Re}$:

$$E = \frac{1}{2}\|\tilde{\mathbf{u}}\|^2 + \frac{1}{2}|\mathcal{P}|\,\|\mathbf{F}\|^2 \leqslant \frac{1}{2}\frac{1}{m^2}\|\boldsymbol{\varepsilon}\mathbf{u}\|^2 + \frac{1}{2}|\mathcal{P}|\,\|\mathbf{F}\|^2 \leqslant \max\left\{\frac{1}{\sigma m^2}, \frac{1}{\rho}\right\}\mathcal{E}.$$

(2) The stability balls δ differ slightly from those derived in [12] for Couette flow. Considering the asymptotic behavior of δ in the limits $\Delta\mathrm{Re} \to 0$ and $m \to 0$, we find

$$\delta^{1/2} \sim \begin{cases} \Delta\mathrm{Re} & \text{in the limit } \Delta\mathrm{Re} \to 0, \\ m^3\sqrt{\alpha\beta} & \text{in the limit } m \to 0. \end{cases}$$

A comparison with the corresponding asymptotic formulas in [12] shows that the asymptotics with respect to $\Delta\mathrm{Re}$ does not depend of the functional (nor of the boundary conditions, cf. [15]); the decay of δ for $m \to 0$, however, is here faster than in [12].

Appendix A

We collect in this appendix some more or less standard inequalities we made use of in the main text. We begin with

Lemma A.1. *Let $n \in \mathbb{N}$ and $a_\nu \geqslant 0$, $b_\nu > 0$ for $1 \leqslant \nu \leqslant n$. Then*

$$\frac{\sum_{\nu=1}^n a_\nu}{\sum_{\nu=1}^n b_\nu} \leqslant \max\left\{\frac{a_\nu}{b_\nu} \;\Big|\; 1 \leqslant \nu \leqslant n\right\} =: M \tag{A.1}$$

and equality holds if and only if $a_\nu = M b_\nu$ for every ν.

Note that inequality (A.1) remains valid for $n \to \infty$.

Frequent use is made of the following Poincaré-type inequalities

$$\|f\| \leqslant \frac{1}{\pi}\|\nabla f\|, \tag{A.2}$$

$$\|\nabla f\| \leqslant \frac{1}{\pi}\|\nabla\nabla f\| = \frac{1}{\pi}\|\Delta f\|, \tag{A.3}$$

which are valid for $\mathcal{P}$-periodic functions f decomposed according to

$$f(x,y,z) = \frac{1}{\sqrt{\mathcal{P}}}\sum_{\boldsymbol{\kappa}\in\mathbb{Z}^2} f_{\boldsymbol{\kappa}}(z)e^{i(\alpha\kappa_1 x + \beta\kappa_2 y)} \tag{A.4}$$

with (at least) $f_{\boldsymbol{\kappa}} \in H^1((-\frac{1}{2},\frac{1}{2}))$ and (weakly) satisfying the boundary conditions $f_{\boldsymbol{\kappa}}(\pm\frac{1}{2}) = 0$, $\boldsymbol{\kappa} \in \mathbb{Z}^2$ (cf. Appendix A in [13]). The inequalities (A.2), (A.3) hold likewise for vector valued functions if each component satisfies such a decomposition.

The next two lemmata provide bounds on the sup-norm $\|\cdot\|_\infty = \operatorname{ess\,sup}|\cdot|$ in terms of the L_2-norm $\|\cdot\|_2 = \|\cdot\|$ in one and three dimensions.

Lemma A.2. *Let $f \in H^1((-\frac{1}{2},\frac{1}{2}))$ with (weakly) $f(-\frac{1}{2}) = 0$. Then*

$$\|f\|_\infty^2 \leqslant 2\|f\|\,\|f'\|. \tag{A.5}$$

Lemma A.3. *Let $f : \mathbb{R}^2 \times [-\frac{1}{2},\frac{1}{2}] \to \mathbb{R}$ be $\mathcal{P}$-periodic and decomposed according to* (A.4) *with $f_{\boldsymbol{\kappa}} \in H^1((-\frac{1}{2},\frac{1}{2}))$ and weakly satisfying the boundary conditions $f_{\boldsymbol{\kappa}}(\pm\frac{1}{2}) = 0$ for $\boldsymbol{\kappa} \in \mathbb{Z}^2 \setminus \{0\}$, $f_0 = \frac{1}{\sqrt{|\mathcal{P}|}}\int_{\mathcal{P}} f(x,y,z)\,dx\,dy = 0$. Then*

$$\|f\|_\infty \leqslant C\left\|(-\Delta_2)^{1/2}\partial_z f\right\|^{1/2}\left\|(-\Delta_2)f\right\|^{1/2} \leqslant \frac{C}{\sqrt{2}}\|\boldsymbol{\delta} f\| \tag{A.6}$$

with $C := 8(\sqrt{2}/m)^{3/2}$, $m := \min\{\alpha,\beta\}$.

A proof of Lemma A.3 can be found in [12].

If f has a nonzero mean value f_0 the inequalities (A.2), (A.5) and (A.6) furnish

$$\|f\|_\infty \leqslant \|\tilde{f}\|_\infty + \|f_0\|_\infty \leqslant \frac{C}{\sqrt{2}}\|\boldsymbol{\delta}\tilde{f}\| + \sqrt{\frac{2}{\pi}}\|f_0'\|, \tag{A.7}$$

where $\tilde{f} = f - f_0$.

The inequalities (A.5)–(A.7) hold likewise for vector valued functions if each component satisfies the appropriate conditions.

References

[1] F.H. Busse, Bounds on the transport of mass and momentum by turbulent flow between parallel plates, Z. Angew. Math. Phys. 20 (1969) 1–14.
[2] F.H. Busse, A property of the energy stability limit for plane parallel shear flow, Arch. Rational Mech. Anal. 47 (1972) 28–35.
[3] O. Dauchot, F. Daviaud, Finite-amplitude perturbation in plane Couette flow, Europhys. Lett. 28 (1994) 225–230.
[4] P.G. Drazin, W.H. Reid, Hydrodynamic Stability, Cambridge Univ. Press, Cambridge, 1981.
[5] J. Flavin, S. Rionero, Qualitative Estimates for Partial Differential Equations. An Introduction, CRC Press, Boca Raton, FL, 1996.
[6] G.P. Galdi, M. Padula, A new approach to the energy theory in the stability of fluid motion, Arch. Rational Mech. Anal. 110 (1990) 187–286.
[7] S. Grossmann, The onset of shear flow turbulence, Rev. Modern Phys. 72 (2000) 603–618.
[8] D.D. Joseph, Nonlinear stability of the Boussinesq equations by the method of energy, Arch. Rational Mech. Anal. 22 (1966) 163–184.
[9] D.D. Joseph, Stability of Fluid Motions, Springer Tracts in Natural Philosophy, vols. 27 and 28, Springer, Berlin, 1976.
[10] R. Kaiser, B. Schmitt, Bounds on the energy stability limit of plane parallel shear flows, Z. Angew. Math. Phys. 52 (2001) 573–596.
[11] R. Kaiser, A. Tilgner, On the generalized energy method for channel flows, in: R. Monaco, M. Pandolfi Bianchi, S. Rionero (Eds.), Proceedings of the 11th Conference on Waves and Stability in Continuous Media, Porto Ercole, 2001, World Scientific, 2002, pp. 259–270.
[12] R. Kaiser, A. Tilgner, W. von Wahl, A generalized energy functional for plane Couette flow, 2004, submitted for publication.
[13] R. Kaiser, L.X. Xu, Nonlinear stability of the rotating Bénard problem, the case $\mathrm{Pr} = 1$, Nonlinear Differential Equations Appl. 5 (1998) 283–307.
[14] W.McF. Orr, The stability or instability of steady motions of a liquid, Part II: A viscous liquid, Proc. Roy. Irish Acad. Sect. A 27 (1907) 69–138.
[15] S. Rionero, G. Mulone, On the nonlinear stability of parallel shear flows, Contin. Mech. Thermodyn. 3 (1991) 1–11.
[16] V.A. Romanov, Stability of plane-parallel Couette flow, Funct. Anal. Appl. 7 (1973) 137–146.
[17] B. Straughan, The Energy Method, Stability and Nonlinear Convection, second ed., Springer, New York, 2004.
[18] B.J. Schmitt, W. von Wahl, Decomposition of solenoidal fields into poloidal fields, toroidal fields and the mean flow. Applications to the Boussinesq-equations, in: J.G. Heywood, K. Masuda, R. Rautmann, S.A. Solonnikov (Eds.), The Navier–Stokes Equations II—Theory and Numerical Methods, Proceedings, Oberwolfach, 1991, in: Lecture Notes in Math., vol. 1530, Springer, Berlin, 1992, pp. 291–305.
[19] W. von Wahl, Necessary and sufficient conditions for the stability of flows of incompressible viscous fluids, Arch. Rational Mech. Anal. 126 (1994) 103–129.